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# SOLUTION OF CERTAIN OPTIMAL CORRECTION PROBLEMS WITH ERROR OF EXECUTION OF THE CONTROL ACTION

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We solve the optimal control problem for the final state of a moving object in the case when there is an error in the execution of the control action and constraints are imposed on the control resources. The stabilization problem for systems in which the hindrance depends upon the magnitude of the control action was examined earlier in [1]. The optimal control problem with errors in the execution of the control action without constraints on the control function was examined in [2]. Certain other optimal correction problems with random perturbations were examined in [3 – 6].

**1. Statement of the problem.** Let the controlled motion of a material point be described by the equation

$$d^2x/dt^2 = u(t) + c_1 |u|^{m/2} \xi(t) + c_2\eta(t), \quad x(t_0) = x_0,\ x^{\cdot}(t_0) = x_0^{\cdot} \quad (1.1)$$

Here $t$ is time, $x$ is a scalar phase coordinate, $u$ is the control function, the quantity $|u|^{m/2}$ has the meaning of the intensity of the perturbations caused by the action of the control, $m$ is some positive number, being a parameter of the problem, $\xi$ and $\eta$ are independent Gaussian noises of unit intensity, $c_1$ and $c_2$ are positive constants.

We are required to find the control satisfying the integral constraint

$$\int_{t_0}^{T} |u|\, dt \leqslant q_0, \qquad q_0 \geqslant 0 \quad (1.2)$$

having the sence of a constraint on the control resources, and minimizing at a finite instant $T$ the mean of the following function of the phase coordinate:

$$J = F[x(T)] \quad (1.3)$$

The function $F$ sets a certain measure of the system's deviation from zero. We assume that it possesses the properties of evenness, nonnegativeness and strict monotonicity and convexity, namely

$$F(x) \geqslant 0,\ F(0) = 0,\ F(x) = F(-x);\ F'(x) > 0,\ F''(x) > 0,\ x > 0 \quad (1.4)$$

The problem formulated models the problem of optimal correction of the lateral deflection on an object in a random force field in the case when additional random perturbations arise, being a consequence of the control force action. We introduce the new variable $y = (T - t)x^{\cdot} + x$. Equation (1.1) takes the form

$$dy/dt = (T - t)\{[u + c_1 |u|^{m/2} \xi] + c_2\eta\}, \quad y(t_0) = y_0 \quad (1.5)$$

Since $y(T) = x(T)$, the functional (1.3) is written in just the same form as before. The equation

$$dy/dt = a(t)\,[u + c\,|u|^{m/2}\,\xi] + b(t)\,\eta, \quad y(t_0) = y_0 \tag{1.6}$$

is a generalization of Eq. (1.5). Here $c$ is a positive constant, $a(t)$, $b(t)$ are smooth monotonically-decreasing functions having the meaning of the intensity of control and of noise, respectively. Without loss of generality we can assume that

$$a(t) > 0, \quad b(t) > 0, \quad 0 \leqslant t \leqslant T; \quad a(T) = b(T) = 0$$

We seek a control satisfying constraint (1.2) and minimizing the mean of functional (1.3).

**2. Fundamental equations.** Let us set up the Bellman-Isaacs equation for problem (1.6) with functional (1.3). We introduce the variable $q$ having the meaning of unspent control resource. The equation and the boundary condition for $q$

$$dq/dt = -\,|u|, \quad q(t_0) = q_0 > 0, \quad q(T) \geqslant 0 \tag{2.1}$$

follows from condition (1.2).

Let $S(t, y, q)$ be the minimum value of the mean of functional $F[y(T)]$, which can be achieved under the initial conditions $t_0 = t$, $y_0 = y$, $q_0 = q$ in problem (1.6), (2.1). The Bellman equation for function $S$ has the form [5, 6]

$$S_\tau = \min_u \left\{ -\,|u|\,S_q + a(\tau)\,uS_y + \frac{1}{2}\,a_1^2(\tau)\,|u|^m\,S_{yy} \right\} + \frac{1}{2}\,b^2(\tau)\,S_{yy} \tag{2.2}$$

$$(a_1(\tau) = ca(\tau))$$

Here $T - t = \tau$ is reverse time, the subscripts on function $S$ denote the taking of the corresponding partial derivatives, $a(\tau)$ and $b(\tau)$ denote the functions $a(t)$ and $b(t)$ with the change of variables $t = T - \tau$. To Eq. (2.2) we add on the initial condition

$$S(0, y, q) = F(y) \tag{2.3}$$

From the problem's statement, the properties (1.4) of function $F$ and the assumption of smoothness of function $S$ in the variable $y$ it follows that

$$S(\tau, y, q) = S(\tau, -y, q), \quad \operatorname{sign} S_y = \operatorname{sign} y \tag{2.4}$$

The first property reflects the evenness of function $S$, the second one follows from the fact that the greater the deflection $|y_0|$ in (1.6) the greater the final deflection $|y(T)|$, other conditions being equal. From (2.4) follows the validity of the boundary condition

$$S_y(\tau, 0, q) = 0 \tag{2.5}$$

Let us carry out the operation of taking the minimum in (2.2) in the region $y \geqslant 0$. It is clear that $u \leqslant 0$ in this region. If $m > 1$, then in order for the minimum of the expression on the right-hand side of (2.2) to be finite, it is necessary that $S_{yy} \geqslant 0$. If, however, $m < 1$, then by the same considerations it is necessary that the condition $S_q + a(\tau)\,S_y \leqslant 0$ be fulfilled; otherwise we get that $S_\tau = -\infty$ and, consequently, $S_t = +\infty$ since $\tau = T - t$. The latter signifies that the average magnitude of the gross error prescribed by function $S$ grows with increasing time $t \leqslant T$ which contradicts the physical sense of the problem. The case $m = 1$ will be analyzed separately.

By computing the minimum of the right-hand side of (2.2), in the region $y \geqslant 0$ we obtain, with due regard to the fact that here $u \leqslant 0$,

$$u = -\left(\frac{1}{m}\right)^{1/(m-1)}\left[\frac{S_q + aS_y}{{}^1/_2 a_1^2 S_{yy}}\right]^{1/(m-1)} \tag{2.6}$$

From (2.6) it follows that in the region $y \geqslant 0$ Eq. (2.2) takes the form

$$S_\tau = \frac{1-m}{m}\left(\frac{1}{m}\right)^{1/(m-1)} \frac{[S_q + aS_y]^{m/(m-1)}}{[{}^1/_2 a_1^2 S_{yy}]^{1/(m-1)}} + \frac{1}{2} b^2 S_{yy} \tag{2.7}$$

Thus, the problem is reduced to seeking the solution of Eq. (2.7), depending on the number $m > 0$ as if on a parameter and satisfying the boundary conditions (2.3), (2.5).

**3. Construction of the solutions.** 3.1. Case $m > 1$. As we have already noted, in this case it is necessary that $S_{yy} \geqslant 0$. Taking properties (1.4) and (2.4) into account, we can reckon that

$$S_{yy} > 0, \qquad y > 0 \tag{3.1}$$

Using formula (2.6) we write Eq. (2.7) in the following form:

$$S_\tau = {}^1/_2 (1-m) \,|\, u \,|^m a_1^2(\tau) S_{yy} + {}^1/_2 b^2(\tau) S_{yy} \tag{3.2}$$

We consider two cases:

a) $b(\tau) \equiv 0$. Assuming that $u < 0$ and taking into account that $m > 1$ and property (3.1), from (3.2) we get that $S_\tau < 0$ when $y > 0$ (we recall that $T - t = \tau$ is reverse time) and, consequently, the value of function $S$ increases monotonically as time $t \leqslant T$ increases. This signifies that in the presence of any control $u \neq 0$ the magnitude of the gross error given by function (1.3) can only grow and, by the same token, any control leads to a worsening of the situation in the sense of criterion (1.3). Therefore, it is necessary that $u = 0$. From Eqs. (1.6) and (2.1) it follows that in this case $y = \text{const}$, $q = \text{const}$ and $S = F(y)$. The case we have considered has a simple physical meaning. In this case the error in the execution of the control action is so large that it surpasses the contribution of the control action itself and, therefore, $u = 0$ is a better method for controlling the system.

b) $b(\tau) \not\equiv 0$. In this case it is necessary that the inequality

$$| u \,| \leqslant \left[\frac{b^2(\tau)}{(m-1)\, a_1^2(\tau)}\right]^{1/m} \tag{3.3}$$

be fulfilled. Indeed, if inequality (3.3) is violated, then from (3.2) it follows that $S_\tau < 0$, and we arrive at the situation described above. Let us show that the optimal control is given by the value

$$u = -\left[\frac{b^2(\tau)}{(m-1)\, a_1^2(\tau)}\right]^{1/m}, \qquad y \geqslant 0 \tag{3.4}$$

Let $S^1$ be the solution of Eq. (3.2) with boundary conditions (2.3), (2.5), corresponding to the value of control $u$ determined by formula (3.4). From (3.2) and (3.4) it follows that $S_\tau^1 = 0$ in region $y \geqslant 0$. By $S^2$ we denote the solution of Eq. (3.2), corresponding to any control satisfying the strict inequality (3.3). From (3.2) and (3.3) it follows that $S_\tau^2 > 0$ in region $y \geqslant 0$. Then we get that

$$\frac{\partial}{\partial \tau}(S^1 - S^2) < 0, \quad y > 0, \quad (S^1 - S^2)\big|_{\tau=0} = 0$$

The last equality follows from boundary condition (2.3). Consequently, $S^1 - S^2 \leqslant 0$, $y \geqslant 0$. This signifies that the value of function $S^1$, corresponding to control (3.4), is always less than the value of function $S^2$, corresponding to any other one from the set of possible controls given by strict inequality (3.3). From (2.1) we obtain that under control (3.4)

$$q(t) = \int_{t_0}^{t} \left[\frac{b_2(\lambda)}{(m-1)a_1^2(\lambda)}\right]^{1/m} d\lambda$$

Let $t^*$ be that instant at which the entire stock of control resource with a control function defined by formula (3.4) is exhausted for given values of $t_0$ and $q_0$, i.e. $q_0 = q(t^*)$. If $t^* \geqslant T$, the control resource stock turns out to be sufficient to carry out control (3.4) up to the final instant $T$. From (3.2) and (3.4) it follows that $S_\tau = 0$ and boundary conditions (2.3), (2.5) are fulfilled. In this case the function $S = F(y)$ is a solution of the problem.

If $t^* < T$, the control resource stock is insufficient to carry out the control (3.4) up to final instant $T$. In this case $S_\tau = 0$ for $0 \leqslant t = T - \tau < t^*$ and $S_\tau = {}^1/_2 b^2 S_{yy}$ when $t^* \leqslant t = T - \tau < T$ for given values of $t_0$, $q_0$. The solution of the problem is determined by the formulas

$$S = \begin{cases} S(\tau^*, y, q) & \tau^* < \tau \leqslant T \\ S(\tau, y, q) & 0 < \tau \leqslant T - \tau^* \end{cases}$$

Here

$$S(\tau, y, q) = [2\sqrt{\pi B(\tau)}]^{-1} \int_0^\infty F(\lambda) \times \tag{3.5}$$

$$\left\{\exp\left[-\frac{(\lambda - y)^2}{4B(\tau)}\right] + \exp\left[-\frac{(\lambda + y)^2}{4B(\tau)}\right]\right\} d\lambda$$

$$B(\tau) = \int_0^\tau b^2(z)\, dz, \qquad \tau^* = T - t^*$$

Thus, in the case being considered the control (3.4) is carried out from the initial instant $t_0$ to the instant $t^*$ depending upon the magnitude $q_0$ of the unspent control resource, and when it is exhausted, the control $u = 0$ and the system is subject only to the external random forces.

3.2. Case $m = 1$. To analyze this case we apply the method used in [3]. We set

$$G = {}^1/_2 a_1^2 S_{yy} - S_q - a(\tau) S_y$$

Then for $m \neq 1$ Eq. (2.7) can be written as

$$S_\tau = \frac{1-m}{2m}\left(\frac{1}{m}\right)^{1/(m-1)} a_1^2 \left(1 - \frac{G}{{}^1/_2 a_1^2 S_{yy}}\right)^{m/(m-1)} S_{yy} + \frac{1}{2} b^2 S_{yy} \tag{3.6}$$

We pass to the limit in (3.6) as $m \to 1$, taking into account here that $S_{yy} > 0$ when $y \neq 0$. If $G < 0$ at some point of region $y > 0$, then the limit of the right-hand side of (3.6) turns out to equal infinity as $m \to 1$, which is meaningless. Therefore, it is necessary to fulfill $G \geqslant 0$ everywhere in the region $y > 0$. Let $D_1$ be that part of the region wherein $G > 0$. In domain $D_1$, passing to the limit in Eq. (3.3) as $m \to 1$, we obtain

$$S_\tau = {}^1/_2 b^2(\tau)\, S_{yy}, \quad {}^1/_2 a_1^2(\tau) S_{yy} - S_q - a(\tau)\, S_y > 0 \qquad (3.7)\ .$$

In this case it follows from (2.6) that $u = 0$ in $D_1$ . The equality

$$ {}^1/_2 a_1^2(\tau)\, S_{yy} - S_q - a(\tau)\, S_y = 0 \qquad (3.8)$$

is fulfilled in the remaining part of the region $y > 0$, which we denote by $D_2$. Here it follows from (2.6) that $u < 0$.

Domains $D_1$ and $D_2$ have the following meaning. In domain $D_1$, wherein $u = 0$, an uncontrolled motion takes place under the action of random forces; in domain $D_2$, as we can conclude from (3.4), by passing to the limit as $m \to 1$, an instantaneous impulse correction takes place ($u$ is a delta function of time). From what has been said it is clear that the determination of the boundary $\Gamma$ separating domains $D_1$ and $D_2$ completely solves the optimal control synthesis problem in this case.

A like problem was examined previously in [5, 6]. As in the cases considered in [5, 6] we can show that the solution of problem (3.7), (3.8) and the determination of the boundary $\Gamma$ of domains $D_1$ and $D_2$ reduce to solving a nonlinear boundary-value problem. We have not succeeded in writing out the solution of problem (3.7), (3.8) in closed form in the general case; however, this can be done in the case when the external random actions are absent, i.e. when $b(\tau) \equiv 0$. In this case domain $D_1$ coincides with the set prescribed by the equations $y = 0,\ q = 0$.

In fact, from the original Eqs. (1.1) we can conclude that if $y = 0$ at some instant $t^*$, then it is sufficient to set $u = 0$ for $t \geqslant t^*$, in order to obtain $y = 0$ by the instant $t = T$. The Bellman function $S = 0$, $t^* \leqslant t \leqslant T$, the phase coordinates of the point, and the control resource stock (see (2.1)) do not vary. We note that the second of conditions (3.7) is not fulfilled here since the passage to the limit in (3.6) was carried out under the assumption that $y \neq 0$. It is clear as well that the set $q = 0$ belongs to domain $D_1$. However, both conditions (3.7) should now be fulfilled on this set.

We prolong the function $S(0, y, q)$ continuously onto negative values of $y$ with continuity preserved

$$S(0, y, q) = F_1(y), \quad y \leqslant 0$$

Here $F_1$ is an unknown function satisfying the condition

$$F_1(0) = F(0) = 0$$

We seek the solution of problem (3.8) with initial condition (2.3) and boundary condition (2.5) in the form

$$S(\tau, y, q) = \int_0^\infty [p(y-\lambda, q; \tau)\, F(\lambda) + p(y+\lambda, q; \tau)\, F_1(\lambda)]\, d\lambda$$

Here

$$p(y-\lambda, q; \tau) = [2a_1(\tau) \sqrt{\pi q}\,]^{-1} \exp\left\{-\frac{[y - a(\tau)\, q - \lambda]^2}{4a_1^2(\tau)\, q}\right\}$$

Using condition (2.5) we obtain the identity

$$\int_0^\infty \left\{F(\lambda) + F_1(\lambda) \exp\left[-\frac{\lambda a(\tau)}{a_1^2(\tau)}\right]\right\} \exp\left\{-\frac{[\lambda - a(\tau)\, q]^2}{4a_1^2(\tau)\, q}\right\} d\lambda = 0$$

Hence, since $a_1 = ca(\tau)$

$$F_1(\lambda) = -F(\lambda) \exp\left[\frac{\lambda}{c^2 a(\tau)}\right]$$

Finally we obtain

$$S = [2a_1 \sqrt{\pi q}]^{-1} \int_0^\infty \left\{ F(\lambda)\, p(y-\lambda, q; \tau) - F(\lambda) \exp\left[\frac{\lambda}{c^2 a(\tau)}\right] p(y+\lambda, q; \tau) \right\} d\lambda \quad (3.9)$$

The function $S$ determined by formula (3.9) serves as the solution of Eq. (3.8) in the region $y > 0$ with boundary conditions (2.3), (2.5). Making the change of variable $z = \lambda + a(\tau)\, q$ in (3.9), we obtain

$$S = [2a_1(\tau) \sqrt{\pi q}]^{-1} \int_{a(\tau)q}^\infty F(z - aq) \left\{ \exp\left[-\frac{(y-z)^2}{4a_1^2(\tau)\, q}\right] - \exp\left[\frac{z - aq}{c^2 a(\tau)}\right] \exp\left[-\frac{(y + z - 2a(\tau)\, q)^2}{4a_1^2(\tau)\, q}\right] \right\} dz$$

Tending $q \to 0$, by virtue of the properties of the fundamental solution of parabolic-type equations (for example, see [7], p. 14) we get that

$$S(\tau, y, q) \to F(y - a(\tau)\, q), \quad q \to 0$$

The function $F(y - a(\tau)\, q)$ satisfies both conditions (3.7) when $q = 0$; moreover, the fulfillment of these conditions is to be understood in this case as the fulfillment of the boundary conditions for the function

$$F(y - a(\tau)\, q) \quad \text{for} \quad q = 0.$$

Thus, the function $S$ constructed by formula (3.9) satisfies all the conditions of problem (3.7), (3.8), whence follows the required assertion that domain $D_1$ is given by the equations $y = 0$, $q = 0$.

3.3. Case $m < 1$. As already noted in the derivation of Eq. (2.7), the necessary condition in this case is the fulfillment of the inequality

$$S_q + a(\tau)\, S_y \leqslant 0, \quad \text{when} \quad y \geqslant 0$$

Using (2.6) we write Eq. (2.7) in the following form:

$$S_\tau = -\left(\frac{1-m}{m}\right) u\, [S_q + a(\tau)\, S_y] + \frac{1}{2} b^2(\tau)\, S_{yy} \quad (3.10)$$

Here $u$ is determined by formula (2.6). We consider two cases:

a) The external random perturbations are absent: $b(\tau) = 0$. Let $u < 0$ and $S_q + a(\tau)\, S_y < 0$. Since $m < 1$, it follows from (3.10) that $S_\tau < 0$, and we arrive at the situation described in case 3. The latter signifies that an optimal control $u \neq 0$ does not exist if $S_q + a(\tau)\, S_y < 0$ and, consequently, only the case $S_q + a(\tau)\, S_y = 0$ can be realized. In order to ascertain what the right-hand side of Eq. (3.10) turns into here, we make the following passage to a limit. We set $S_q + a(\tau)\, S_y = \delta < 0$ and we tend $\delta \to -0$. Taking into account that $m < 1$, from (2.7) we get that

$$\lim_{\delta \to -0} \left(\frac{1-m}{m}\right)\left(\frac{1}{m}\right)^{1/m-1} \frac{(\delta)^{m/m-1}}{[{}^1/_2\, a_1^2(\tau)\, S_{yy}]^{1/m-1}} = +\infty$$

i.e. Eq. (3.10) takes the form $S_\tau = +\infty$. This signifies that an instantaneous impulse correction takes place ($u$ is a delta function of time), as a result of which the phase point is displaced along the direction of the characteristics $\eta = y - a(\tau)\, q$, $\eta = \text{const}$, of the equation $S_q + a(\tau)\, S_y = 0$. In this case the solution of the problem coincides with the solution of the corresponding deterministic problem, namely, the problem without random perturbations, and is determined by the formula

$$S(\tau, y, q) = \begin{cases} 0, & y \leqslant a(\tau) q \\ F(y - a(\tau) q), & y > a(\tau) q \end{cases}$$

We recall that $a(\tau)$ is a monotonically increasing function and $a(0) = 0$. Thus, in this case the impulse correction takes place at the initial instant, as a result of which either the phase point hits onto the set $y = 0$ or the control resource stock is completely exhausted.

b) $b(\tau) \neq 0$. We write Eq. (2.7) in form (3.2) and (3.10). Then it is necessary that

$$-\frac{(1-m)}{m} u [S_q + a(\tau) S_y] = \left(\frac{1-m}{2}\right) |u|^m a_1^2(\tau) S_{yy} \qquad (3.11)$$

If $S_q + a(\tau) S_y < 0$, then by virtue of the fact that $S_{yy} > 0$ for $y > 0$ and $u < 0$, $m < 1$, we get that equality (3.11) can be fulfilled only for $u = 0$. Consequently, in that part of region $y > 0$, wherein $S_q + a(\tau) S_y < 0$, it is necessary that $u = 0$ and that the equation

$$S_\tau = {}^1/_2 b^2(\tau) S_{yy} \qquad (3.12)$$

be fulfilled. We denote this set by $\Omega_1$. By $\Omega_2$ we denote that remaining part of region $y > 0$, wherein

$$S_q + a(\tau) S_y = 0 \qquad (3.13)$$

The problem is reduced to seeking the boundary $\Gamma$ of domains $\Omega_1$ and $\Omega_2$ and to solving Eqs. (3.12) in domain $\Omega_1$ and (3.13) in domain $\Omega_2$. But this problem has been examined in [5, 6] where it was shown that domains $\Omega_1$ and $\Omega_2$ have the following meaning. In domain $\Omega_1$, wherein $u = 0$, an uncontrolled motion takes place under the action of the random forces; in domain $\Omega_2$, $u < 0$ and an instantaneous impulse correction takes place ($u$ is a delta function of time), as a result of which the phase point is moved along the corresponding characteristics $\eta = y - a(\tau) q$, $\eta = \text{const}$ of Eq. (3.13). As a result of the correction either the point turns out to be on boundary $\Gamma$ of domains $\Omega_1$ and $\Omega_2$ or the control resource stock is completely exhausted. It is clear that the determination of boundary $\Gamma$ completely solves the optimal control synthesis problem in this case. As was shown in [5] the determination of boundary $\Gamma$ reduces to the solving of a nonlinear boundary-value problem. The position of boundary $\Gamma$ when $a(\tau) = b(\tau) = T - t$, i.e. in the case corresponding to problem (1.5), was found numerically in [5] by means of introducing self-similar variables.

Thus when $m < 1$, the solution of the problem posed coincides with the solution of the corresponding problem without error in execution of the control action.

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# MINIMIZATION OF THE INTEGRAL ESTIMATE OF THE KINETIC ENERGY OF A HARMONIC OSCILLATOR BY AN IMPULSE CONTROL

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The kinetic energy of the transient response of a harmonic oscillator is minimized by actions with a bounded current total momentum. Such a problem arises, for example, when choosing a mass flow program minimizing the kinetic energy of transfer of a satellite into a circular orbit by a reactive force having the direction of the Earth's gravitational force (see [1], p. 32). It is shown that the optimal control contains an impulse component. This leads to the violation of the optimality principle for extremals. Therefore, the synthesis procedure is based on the analysis of an auxiliary variational problem under the usual constraints on the control [2].

**1. Statement of the problem and its reduction.** Let the plant be described by the differential equation of a controlled harmonic oscillator

$$x^{\cdot\cdot} + \omega^2 x = ku \quad (\omega, k \neq 0) \tag{1.1}$$

(where $u$ is the control)

$$u(t) = 0\ (t < 0);\ |\nu[u](t)| \leqslant 1,\ \nu[u](t) = \int_{-\infty}^{t} u(\tau)\,d\tau \tag{1.2}$$

Here $\nu[u](t)$ is a quantity proportional to the current value of the total momentum of the control force. We examine the autput $x[x_0, x_0^{\cdot}; u](t)$ of plant (1.1), corresponding to the initial conditions

$$x[x_0, x_0^{\cdot}; u](0) = x_0, \qquad x^{\cdot}[x_0, x_0^{\cdot}; u](0) = x_0^{\cdot} \tag{1.3}$$

and to some program $u(t)$, subject to requirements (1.2). We define the control's performance index

$$\Delta[x_0, x_0, u] = \int_0^{\infty} \{x^{\cdot}[x_0, x_0^{\cdot}, u](t)\}^2\,dt \tag{1.4}$$